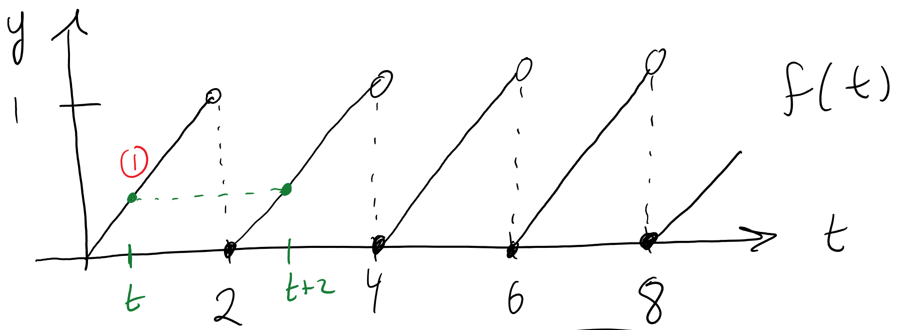
Laplace Transform of a Periodic Function

Definition 5.2. Let $f(t)$ be a piecewise continuous periodic function defined on $0 \le t < \infty$ with period T. Then the Laplace transform of $f(t)$ is the function defined by

$$\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\, dt}{1 - e^{-sT}}, \qquad s > 0 \tag{5.3}$$

Laplace transform over one period

Ex!



(1) $f(t) = \frac{1}{2} t \qquad 0 \le t < 2$

$t+2) = f(t)$ Period $T = 2$

$$\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\, dt}{1 - e^{-sT}}$$

$$\int_0^2 \frac{1}{2} t e^{-st}\, dt = -\frac{1}{2} t \frac{e^{-st}}{s} - \frac{1}{2s^2}\int_0^2 e^{-st}(-s\,dt)$$

$u = \frac{1}{2} t \qquad dv = e^{-st}\, dt$

$du = \frac{1}{2} dt \qquad v = -\frac{e^{-st}}{s}$

$u = -st$

$du = -s\, dt$

$$= -\frac{t e^{-st}}{2s} - \frac{e^{-st}}{2s^2}\Bigg|_0^2$$

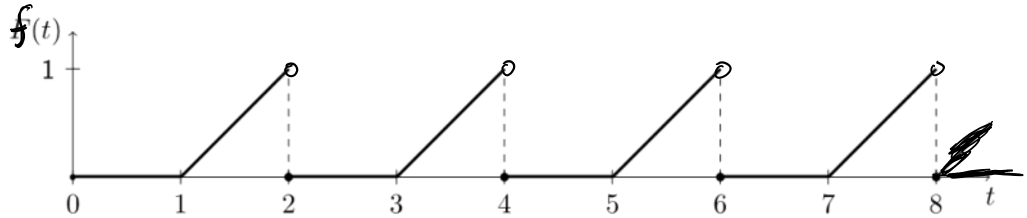
$$= \left(-\frac{2se^{-2s}}{2s^2} - \frac{e^{-2s}}{2s^2}\right) - \left(-\frac{1}{2s^2}\right)$$

$$= \left(-\frac{2se}{2s^2} - \frac{e}{2s^2}\right) - \left(-\frac{1}{2s^2}\right)$$

$$= \frac{-2s\,e^{-2s} - e^{-2s} + 1}{2s^2}$$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{-2s\,e^{-2s} - e^{-s} + 1}{2s^2\left(1 - e^{-2s}\right)}}$$

Example 5.4.2. Find the Laplace transform of the function whose graph is shown



$$f(t) = \begin{cases} 0 & 0 \le t < 1 \\ t-1 & 1 \le t < 2 \end{cases} \qquad T = 2$$

$$\mathcal{L}\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1 - e^{-sT}}$$

$$\int_0^2 f(t)\,e^{-st}\,dt = \int_1^2 (t-1)\,e^{-st}\,dt$$

$u = t-1 \quad dv = e^{-st}\,dt$

$$= -\frac{(t-1)}{s} e^{-st} + \frac{1}{s}\int_1^2 e^{-st}\,dt$$

$$u = t-1 \qquad dv = e^{-st}\,dt$$
$$du = dt \qquad v = -\frac{e^{-st}}{s}$$

$$= -\frac{(t-1)}{s}e^{-st} + \frac{1}{s}\int_1 e^{-st}\,dt$$

$$= -\frac{(t-1)}{s}e^{-st} - \frac{1}{s^2}e^{-st}\Big|_1^2$$

$$= \left(-\frac{s}{s^2}e^{-2s} - \frac{1}{s^2}e^{-2s}\right) - \left(-\frac{1}{s^2}e^{-s}\right)$$

$$= \frac{-s e^{-2s} - e^{-2s} + e^{-s}}{s^2}$$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{-s e^{-2s} - e^{-2s} + e^{-s}}{s^2(1-e^{-2s})}}$$